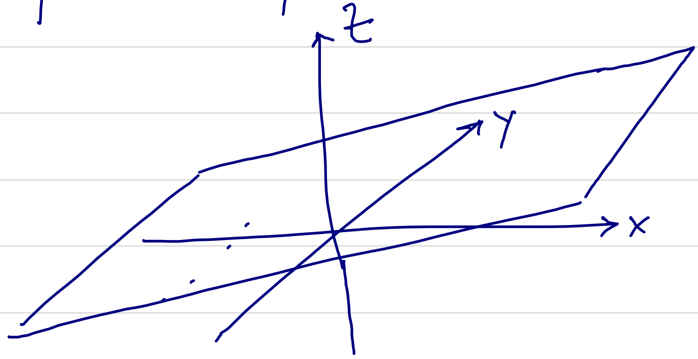
USE GAUSS JORAN (Sec 8.3)

System of Linear Equations in three variables

Just as the equation $Ax + By = C$ represents a line in the plane, the equation $Ax + By + Cz = D$ represents a plane in three dimensional space.

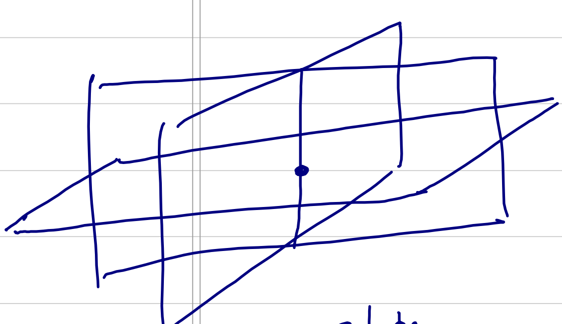


A system of linear equations consists of two or three linear equations.

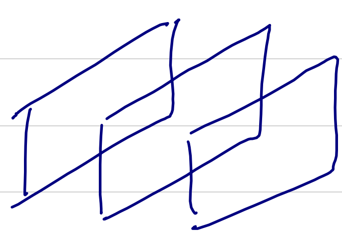
$$A_1x + B_1y + C_1z = D_1 \qquad (i)$$
$$A_2x + B_2y + C_2z = D_2 \qquad (ii)$$
$$A_3x + B_3y + C_3z = D_3 \qquad (iii)$$

Ques. What are the possibilities for the number of solutions?

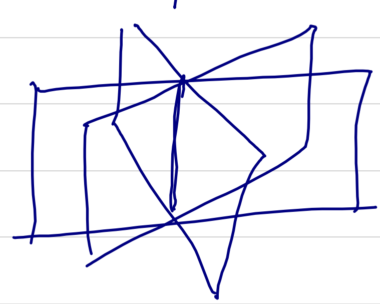
Ans. We can do this graphically. Note that the solution is the intersection of two or three planes in space



one solutn



parallel
no solutns.



Infinitely many solutions

Solve

$$2x + y + 8z = -1 \qquad (i)$$
$$x - y + z = -2 \qquad (ii)$$
$$3x - 2y - 2z = 2 \qquad (iii)$$

<u>Method</u>: 1> Reduce the system into two equations in two variables by eliminating one of the variables.

2> Solve the resulting system of two linear equations in two variables

3> Substitute the values of step 2 into any one equation and solve for the remaining variable.

<u>Soln</u>. It's easy to eliminate y from eq^n (i) and (ii). Adding equations (i) and (ii) we get

$$\begin{array}{rrrl} & 2x + y + 8z & = & -1 \\ + & x - y + z & = & -2 \\ \hline & 3x \quad + 9z & = & -3 \quad (iv) \end{array}$$

Multiplying equation (ii) by -2 and adding to equation (iii) we get

$$\begin{array}{rrrl} & -2x + 2y - 2z & = & 4 \\ + & 3x - 2y - 2z & = & 2 \\ \hline & x \quad -4z & = & 6 \quad (v) \end{array}$$

Now we can solve equations (iv) and (v) for x and z.

Multiplying equation (v) by -3 and adding to equation (iv) we have

$$\begin{array}{rrcl} & -3x + 12z & = & -18 \\ + & 3x + 9z & = & -3 \\ \hline & 21z & = & -21 \end{array}$$

$$\Rightarrow z = -1$$

Substituting this value into equation (v)

$$x - 4(-1) = 6$$

or, $x + 4 = 6$

or, $x = 2$

Substituting $x = 2$ and $z = -1$ into equation (ii) (this is arbitrary)

$$2 - y + (-1) = -2$$

or, $1 - y = -2$

or, $y = 3$

$\therefore$ $x = 2$, $y = 3$, $z = -1$ is the solution.

Exercises:

Solve
$$2x - y + 3z = -1$$
$$x + y - z = 0$$
$$3x + 3y - 2z = 1$$

Dependent system (Infinite solns.)

$$2x + y - z = 4 \quad (i)$$
$$x + y = 2 \quad (ii)$$
$$3x + 2y - z = 6 \quad (iii)$$

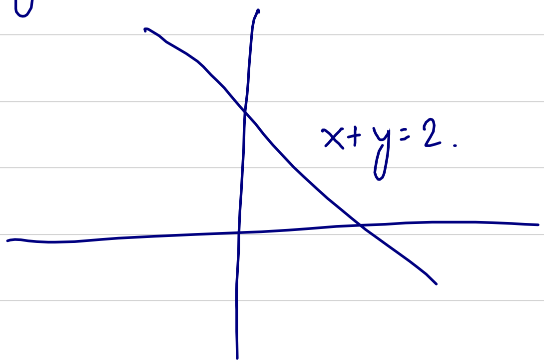
Soln. We can eliminate z easily. Multiplying equation (i) by -1 and adding to equation (iii).

$$\begin{array}{rl} & -2x - y + z = -4 \quad (i) \\ + & 3x + 2y - z = 6 \quad (iii) \\ \hline & x + y = 2 \quad (iv) \end{array}$$

Now we can solve

$$x + y = 2 \quad (iii)$$
$$x + y = 2 \quad (iv)$$

These are the same equations. So any solution of one equation is a solution of this system. Usually in this step we would get $x =$ something and $y =$ other thing but we got infinitely many solutions.



To find the general solution we will fix a value of x. (You can x, y or z in this step).
So let $x = t$ where t is a real numbers.
By equation (ii),

$$y = 2 - x$$
$$= 2 - t.$$

Substituting these values of x and y into equation (i) we get

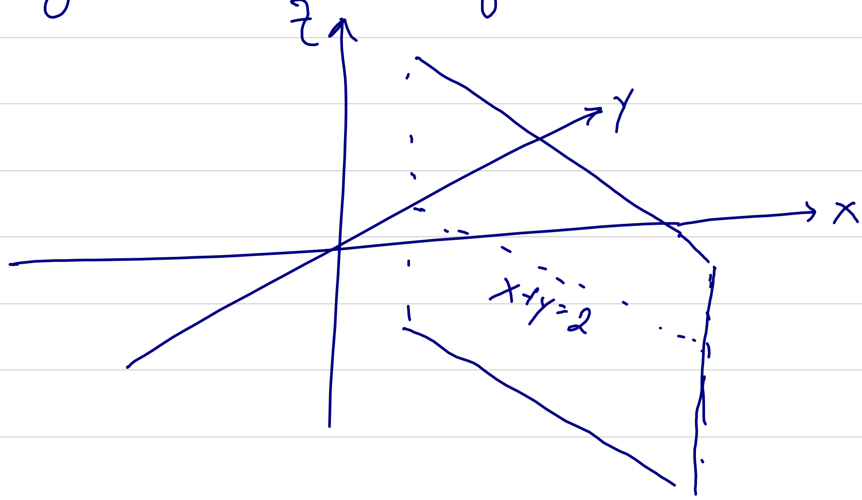
$$2t + (2-t) - z = 4$$
$$t + 2 - z = 4$$
$$\text{or, } z = t - 2.$$

Therefore the general solution is given by $(t, 2-t, t-2)$.

For ex:

if $x = 1$, then $t = 1$ and
$y = 2 - t = 2 - 1 = 1$
$z = t - 2 = 1 - 2 = -1$

$(1, 1, -1)$ is a soln. There are infinitely many solutions : one for each value of x.



Incosistent System

$$x + 2y - z = 3 \quad (i)$$
$$2x - y + 2z = -1 \quad (ii)$$
$$-2x - 4y + 2z = 5 \quad (iii)$$

Soln. Adding eq (ii) and (iii):

$$\begin{array}{r} 2x - y + 2z = -1 \\ -2x - 4y + 2z = 5 \\ \hline -5y + 4z = 4 \end{array} \quad (iv)$$

multiplying equation (i) by 2 and adding to equation (iii):

$$\begin{array}{r} 2x + 4y - 2z = 6 \\ -2x - 4y + 2z = 5 \\ \hline 0 = 11 \end{array}$$

This is impossible. So there are no solutions.

Omit Solving two linear equations in three variables.

Solve

$$x - y + z = 7 \qquad (i)$$
$$x + y + 2z = 2 \qquad (ii)$$

Soln. Try to eliminate one variable.

Adding equations (i) and (ii) we get

$$\begin{array}{r} x - y + z = 7 \\ x + y + 2z = 2 \\ \hline 2x \quad + 3z = 9 \end{array} \qquad (iii)$$

Now fix either x or z. I will fix x.

Let $x = t$ where t is a real number.

Then from equation (iii),

$$2t + 3z = 9$$
or, $3z = 9 - 2t$

or, $z = \dfrac{9-2t}{3}$.

Now plug $x = t$, $z = \dfrac{9-2t}{3}$ into equation (i):

(You could plug into eq. (ii))

$$t - y + \frac{9-2t}{3} = 7$$

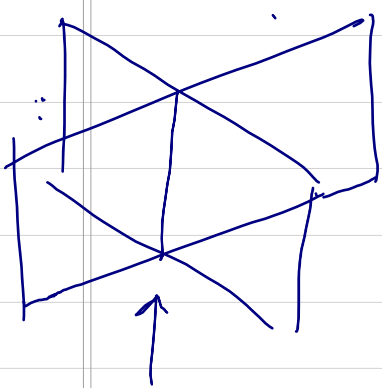
$$y = t + \frac{9-2t}{3} + 7$$
$$= \frac{3t + 9 - 2t + 21}{3}$$
$$= \frac{t + 30}{3}$$

Thus, the solution of the given system is

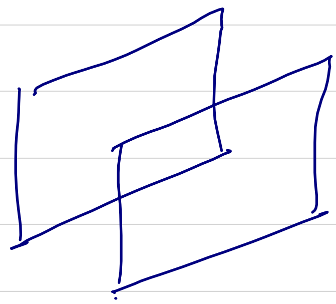
$$\left(t, \frac{t+30}{3}, \frac{9-2t}{3}\right) \quad \square.$$

Note that equations (i) and (ii) represent two planes in space.

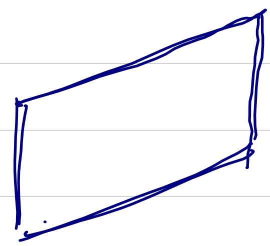
Ques. What are the possibilities?



line is the solution set



parallel
no solution



Same plane
the plane is the solution set

$2x - y + 3z = -1$ (1)

$x + y - z = 0$ (2)

$3x + 3y - 2z = 1$ (3)

Adding eqns (1) and (2)

$$\begin{aligned} 2x - y + 3z &= -1 \\ x + y - z &= 0 \\ \hline 3x \quad - 4z &= 1 \end{aligned}$$ (4)

Multiplying eqn (1) by 3 and adding to eqn (3)

$$\begin{aligned} 6x - 3y + 9z &= -3 \\ 3x + 3y - 2z &= 1 \\ \hline 9x \quad - 7z &= -2 \end{aligned}$$ (5)

Multiply eqn (4) by -3 and adding to eqn (5)

$$\begin{aligned} \cancel{-9x} + 12z &= -3 \\ \cancel{9x} - 7z &= -2 \\ \hline 5z &= -5 \\ z &= -1 \end{aligned}$$

$3x - 4(-1) = 1$

$3x + 4 = 1$

$3x = -3$

$x = -1$

Plugging

$-1 + y + 1 = 0$

$y = 0$

$(-1, 0, -1)$